Gain Indexing Schemes for Low-Thrust Perturbation Guidance

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Two perturbation guidance schemes, time-to-go guidance and minimum-distance guidance, are re-examined in the context of a low-thrust orbit transfer problem. The two schemes, which use different techniques for indexing feedback gains, are shown to be comparable in performance. Both schemes are found to produce terminal state errors that are orders of magnitude smaller than those obtained in several previous studies. Various small modifications or enhancements of the algorithms are thought to account for a portion of the dramatic improvement in results. The problem investigated is a hypothetical Earth to Mars orbit transfer with six state variables, two control variables, and six terminal state constraints.

Introduction

NUMEROUS techniques for controlling the flight path of an aerospace vehicle have been proposed during the past 20 years. Among these is the neighboring optimum control scheme of Breakwell et al.,[1] in which the logic for controlling deviations about a nominal optimal trajectory is determined by minimizing a second-order expansion of some performance index, subject to first-order expansions of the appropriate differential and terminal constraints. A linear state perturbation feedback control law, with time-varying gains, results from this approach. One difficulty with the implementation of such a scheme in problems with an unspecified final time is the determination of which feedback gains to use at any given time. The selection of gains according to current time can cause serious problems, if the final time on the perturbed trajectory is different from the final time on the nominal trajectory, particularly if the former is greater than the latter.

To circumvent these gain indexing difficulties, Kelley[2] and Speyer and Bryson[3] proposed the use of performance index-to-go and time-to-go, respectively, rather than current time, for gain indexing. (These ideas appear to have originated with an unpublished suggestion by J.C. Dunn.[2]) Speyer and Bryson applied the time-to-go guidance technique to a re-entry problem with three state variables, one control variable, and two terminal state constraints, and found that this approach works extremely well (and much better than a current-time gain indexing approach). Wood[4,5] applied the time-to-go guidance technique to a low-thrust orbit transfer problem with three state variables, one control variable, and three terminal state constraints, and again found that this indexing technique works extremely well. This orbit transfer problem included an explicit time dependence in the system dynamics, which the re-entry problem did not, and thus verified portions of the time-to-go guidance logic that had been suggested, but not tested, by Speyer and Bryson.

Powers[6,7] proposed an alternative gain indexing scheme, referred to as min-distance guidance, and argued that it should perform better than time-to-go guidance. He considered Zermelo's problem with one state variable, one control variable, and one terminal state constraint.[7] He did not apply the guidance techniques over an entire trajectory, as was done in Refs. 3-5, but compared the control corrections calculated by the two schemes immediately after a large state perturbation. He found that, for a suitable choice of weighting coefficients, the control correction calculated by the min-distance scheme was considerably more accurate than that calculated by the time-to-go scheme. Hart[8] compared min-distance and time-to-go guidance schemes, in the context of a low-thrust orbit transfer problem with six state variables, two control variables, and six terminal state constraints. He also found that min-distance guidance performs better than time-to-go guidance. However, inspection of his numerical results indicates that neither scheme performed very well, with state deviations frequently increasing, rather than decreasing, with time. Lattimore[9] subsequently investigated the behavior of the min-distance guidance approach as a function of the guidance loop closure rate. His terminal errors were typically larger than Hart's. Stoker[10] re-examined the orbit transfer problem of Hart and Lattimore and obtained terminal errors that were generally smaller than those in Refs. 8 and 9, but that were still quite large. He also found the performances of the two schemes to be essentially indistinguishable.

It was to resolve the conflicting evidence regarding the relative and absolute merits of the time-to-go and min-distance guidance schemes that the present study was undertaken. To facilitate comparisons with earlier work, the orbit transfer problem of Refs. 8-10 was examined. It should be stressed at this point that, while the remainder of this paper deals specifically with a low-thrust orbit transfer problem, the applications of these guidance techniques in the aerospace field are much broader than this. It should also be pointed out that the orbit transfer problem investigated is not intended to be fully realistic in all respects. It is merely the most sophisticated problem to which these guidance techniques have been applied and about which information is readily available. The problem is also a logical generalization of the planar orbit transfer problem considered in Refs. 4 and 5.

Earth to Mars Orbit Transfer Problem

The problem considered here is that of transferring a vehicle from Earth to Mars, using a constant, low-thrust rocket engine. The gravity fields of Earth and Mars are ignored. The only forces assumed to be acting on the spacecraft are the central force field of the sun and the rocket thrust force. These assumptions remove complexity from the problem and are unlikely to affect the relative performance of the guidance algorithms.

The inertially fixed, heliocentric coordinate system used to describe the problem has the following orthogonal position

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coordinates: x_4 in the direction of the ascending node of Mars at a reference date, x_5 in the ecliptic plane, and x_6 parallel to the Earth's orbital angular momentum vector (see Fig. 1 of Ref. 11 or Fig. 1 of Ref. 12).

The equations of motion of the spacecraft are

$$\dot{x}_1 = \frac{-\mu x_4}{r^3} + \frac{\beta v_{\text{ex}}}{m_0 - \beta t} \cos u_1 \cos u_2 \tag{1}$$

$$\dot{x}_2 = \frac{-\mu x_5}{r^3} + \frac{\beta v_{\text{ex}}}{m_0 - \beta t} \cos u_1 \sin u_2 \tag{2}$$

$$\dot{x}_3 = \frac{-\mu x_6}{r^3} + \frac{\beta v_{\text{ex}}}{m_0 - \beta t} \sin u_1 \tag{3}$$

$$\dot{x}_{i+3} = x_i, \qquad i = 1,2,3 \tag{4}$$

where

$$r = \sqrt{x_4^2 + x_5^2 + x_6^2} \tag{5}$$

and

x_1, x_2, x_3 = heliocentric velocity coordinates
μ = gravitational parameter of the sun
u_1, u_2 = polar and azimuthal inertial thrust vector control angles (see Fig. 1 of Ref. 11)
β = mass flow rate (constant)
v_{ex} = rocket exhaust velocity (constant)
m_0 = initial spacecraft gross mass
t = time ($t = 0$ at launch)

A dot over a quantity denotes differentiation with respect to time.

The initial conditions are meant to correspond to the Earth's state on the hypothetical launch date, 1200 Jan. 9, 1982, and are given in Table 1 of Ref. 11 and Table 1 of Ref. 12. The terminal constraints reflect a rendezvous with the center of Mars,

$$x_i(t_f) = x_{M_i}(t_f), \qquad i = 1,\ldots,6 \tag{6}$$

where $x_{M_i}(t)$ denote the components of the heliocentric velocity and position vectors of Mars at time t. Orbital elements for Mars and other constants associated with this problem are given in Table 2 of Ref. 11 and Table 4 of Ref. 12.

Final mass is to be maximized in this problem. This is the same as minimizing propellant consumption, assuming a fixed initial mass. Therefore, the performance index may be expressed as

$$J = \int_0^{t_f} \beta \, \mathrm{d}t \tag{7}$$

Since β is, by assumption, constrained to be a constant, minimizing the expression in Eq. (7) is equivalent to minimizing final time.

The orbit transfer problem stated above is exactly the same as in Refs. 8-10. If the orbits of Earth and Mars are made circular and coplanar and if three terminal state constraints (on out-of-plane position, out-of-plane velocity, and heliocentric angle) are deleted, the problem becomes essentially that of Refs. 4 and 5.

Determination of the Nominal Optimal Trajectory

If Eqs. (1-4) and (6) and the initial conditions are written in vector form,

$$\dot{x} = f[x(t), u(t), t], \qquad 0 \le t \le t_f \tag{8}$$

$$x(0) = x_0 \tag{9}$$

$$\psi[x(t_f), t_f] = 0 \tag{10}$$

it can be shown[13] that necessary conditions to minimize a performance index of the form

$$J = \phi[x(t_f), t_f] + \int_0^{t_f} L[x(t), u(t), t] \, \mathrm{d}t \tag{11}$$

are given by

$$\dot{\lambda}^T = \frac{-\partial H}{\partial x}, \qquad 0 \le t \le t_f \tag{12}$$

$$\lambda^T(t_f) = \left[\frac{\partial \Phi}{\partial x}\right]_{t=t_f} \tag{13}$$

$$\Omega = \left[\frac{\partial \Phi}{\partial t} + \frac{\partial \Phi}{\partial x} f + L\right]_{t=t_f} = 0 \tag{14}$$

$$\frac{\partial H}{\partial u} = 0, \qquad 0 \le t \le t_f \tag{15}$$

$$\frac{\partial}{\partial u}\left[\left(\frac{\partial H}{\partial u}\right)^T\right] \text{ positive semidefinite}, \quad 0 \le t \le t_f \tag{16}$$

and Eqs. (8-10), where $\lambda(t)$ and ν are vectors of Lagrange multipliers and

$$H = L(x,u,t) + \lambda^T(t) f(x,u,t) \tag{17}$$

$$\Phi = \Phi[x(t_f), t_f] + \nu^T \psi[x(t_f), t_f] \tag{18}$$

x, u, λ and f are all column vectors. The derivative of a scalar quantity with respect to a column vector is defined to be a row vector. These necessary conditions form a two-point boundary value problem (TPBVP), which must be solved numerically.

A computer program was written to solve this TPBVP using the backward sweep method,[13] a type of neighboring extremal algorithm. A high-fidelity integrator, DODE,[14] was used, and considerable care was taken to insure high accuracy in the solution for the nominal trajectory and the associated feedback gain information. By way of contrast, the converged solution [$x(0)$, $\lambda(0)$, t_f] of the current study (see Table 1 of Ref. 11 or Table 1 of Ref. 12), when integrated forward to t_f,

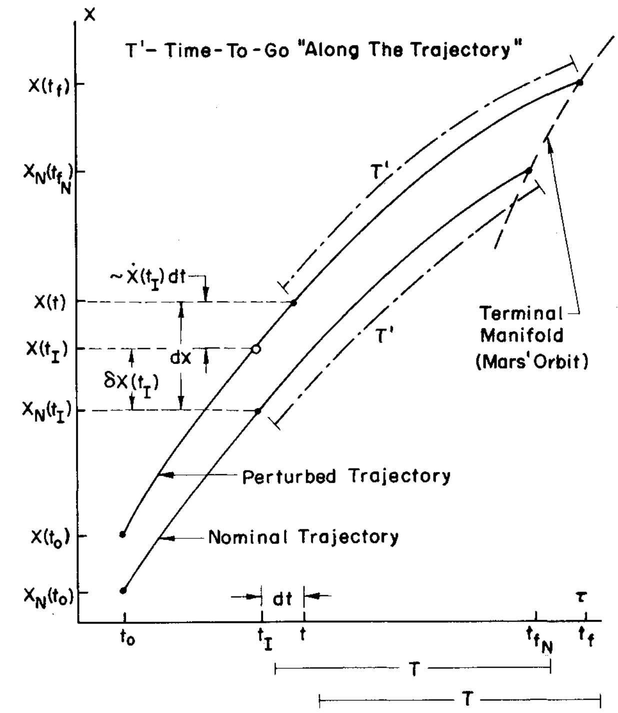

Fig. 1 One state component representation of time-to-go indexing.

differs from Mars' state by less than 0.004 km in position, while that of the previous studies[8-10] appears to differ by 75,000 km. The contrast in velocity errors is similar. It is believed that this greater accuracy in the nominal trajectory and in the feedback gain information may play a part in the improved performance of the guidance schemes obtained in this study.

An ecliptic plane projection of the nominal trajectory is shown in Fig. 2 of Ref. 11 and Fig. 56 of Ref. 12. Further information about the optimization of the nominal trajectory may be found in Ref. 12.

Neighboring Optimum Guidance

The actual trajectory flown by an aerospace vehicle is never precisely the same as a precalculated nominal trajectory, due to systematic and random errors in the dynamical model of the vehicle motion and in the models of the measurements from which the vehicle state is inferred. One technique for handling deviations from a nominal optimal trajectory is to recompute new optimal trajectories in flight as deviations are sensed. In some applications, the time, cost, and information storage requirements for such an approach are prohibitive. A less demanding approach involves the solution of an approximate form of the optimization problem, with the performance index expanded to second order in state and control perturbations about the nominal trajectory and the state dynamics and terminal constraints expanded to first order.[1,13] This procedure approximates the original nonlinear optimal control problem by a linear-quadratic optimal control problem (the accessory minimum problem), the solution to which is characterized by a linear state perturbation feedback control law. The feedback gains are related to the nominal trajectory and may be calculated in advance and stored for use in flight, making the real-time control computations relatively simple.

If we define

$$\delta x(t) = x(t) - x_N(t) \tag{19}$$

$$\delta u(t) = u(t) - u_N(t) \tag{20}$$

$$\delta\lambda(t) = \lambda(t) - \lambda_N(t) \tag{21}$$

$$\mathrm{d}\bar{\nu} = \bar{\nu} - \bar{\nu}_N \tag{22}$$

$$\bar{\nu} = \begin{bmatrix} \nu \\ t_f \end{bmatrix} \tag{23}$$

with the subscript N referring to the nominal trajectory, then it turns out[3,5,13] that

$$\delta u(t) \cong -H_{uu}^{-1}[H_{ux}\delta x(t) + f_u^T \delta\lambda(t)] \tag{24}$$

and

$$\begin{bmatrix} \delta\lambda(t) \\ 0 \end{bmatrix} \cong \begin{bmatrix} S(t) & R(t) \\ R^T(t) & Q(t) \end{bmatrix} \begin{bmatrix} \delta x(t) \\ \mathrm{d}\nu \end{bmatrix} \tag{25}$$

or

$$\begin{bmatrix} \delta\lambda(t) \\ -\mathrm{d}\bar{\nu} \end{bmatrix} \cong \begin{bmatrix} S_*(t) & R_*(t) \\ R_*^T(t) & Q_*(t) \end{bmatrix} \begin{bmatrix} \delta x(t) \\ 0 \end{bmatrix} \tag{26}$$

so that

$$\delta u(t) \cong -H_{uu}^{-1}[(H_{ux} + f_u^T S)\delta x(t) + f_u^T R \mathrm{d}\bar{\nu}] \tag{27}$$

or

$$\delta u(t) \cong -H_{uu}^{-1}(H_{ux} + f_u^T S_*)\delta x(t) \tag{28}$$

where

$$S_* = S - RQ^{-1}R^T \tag{29}$$

$$R_* = RQ^{-1} \tag{30}$$

$$Q_* = -Q^{-1} \tag{31}$$

$$\left.\begin{aligned} \dot{S} &= -A^T S - SA + SBS - C \qquad (32)\\ \dot{R} &= -(A^T - SB)R \qquad (33)\\ \dot{Q} &= R^T B R \qquad (34)\end{aligned}\right\} 0 \le t \le t_f$$

$$S(t_f) = [\Phi_{xx}]_{t=t_f} \tag{35}$$

$$R(t_f) = \left[\left(\frac{\partial\psi}{\partial x}\right)^T \quad \left(\frac{\partial\Omega}{\partial x}\right)^T\right]_{t=t_f} \tag{36}$$

$$Q(t_f) = \begin{bmatrix} 0 & \dfrac{\mathrm{d}\psi}{\mathrm{d}t} \\ \left(\dfrac{\mathrm{d}\psi}{\mathrm{d}t}\right)^T & \dfrac{\mathrm{d}\Omega}{\mathrm{d}t} \end{bmatrix}_{t=t_f} \tag{37}$$

$$A(t) = f_x - f_u H_{uu}^{-1} H_{ux} \tag{38}$$

$$B(t) = f_u H_{uu}^{-1} f_u^T \tag{39}$$

$$C(t) = H_{xx} - H_{xu} H_{uu}^{-1} H_{ux} \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) \equiv \frac{\partial}{\partial t}(\cdot) + \frac{\partial}{\partial x}(\cdot)\dot{x} + \frac{\partial}{\partial u}(\cdot)\dot{u} \tag{41}$$

$$H_{ux} \equiv \frac{\partial}{\partial x}(H_u^T) \tag{42}$$

Equations (25-28) are easily generalized to situations in which the terminal constraints are changed in flight.[3,5,13]

The optimal control at time t is thus given by

$$u(t) = u_N(t) + \delta u(t) \tag{43}$$

where $\delta u(t)$ is determined from Eq. (28). It sometimes happens that the matrices $S(t)$, $R(t)$, and $Q(t)$ diverge somewhere between t_f and zero.[3,5] In this case, $S(t)$, $R(t)$, and $Q(t)$ should be integrated backward from t_f until $Q(t)$ is sufficiently nonsingular to be inverted accurately, S_*, R_*, and Q_* then evaluated using Eqs. (29-31), and the backward integration resumed using the starred matrices, which satisfy Eqs. (32-34) as well.[3,5] (Only one column of R_* and none of Q_* is needed in the subsequent development.)

Time-to-Go Indexing

Time-to-go (TTG) indexing[3] is based upon the idea that satisfaction of the terminal constraints and minimization of the performance index should be much improved, as compared with the current-time indexing scheme described above, if the gain information is taken from the nominal trajectory at an index time such that the time-to-go on the nominal trajectory is the same as the time-to-go on the perturbed trajectory (see Fig. 1). In other words, the index time t_I should be selected such that

$$t_f - t = t_{f_N} - t_I \tag{44}$$

Of course, the actual value of t_f is not known until the flight is over, so that Eq. (44) cannot be used directly to determine t_I. However, the shift in t_f may be approximated by making use of the bottom row of Eq. (26), which yields

$$\mathrm{d}t_f \cong -m_*^T(t)\delta x(t) \tag{45}$$

where m_* is the last column of the matrix R_*.

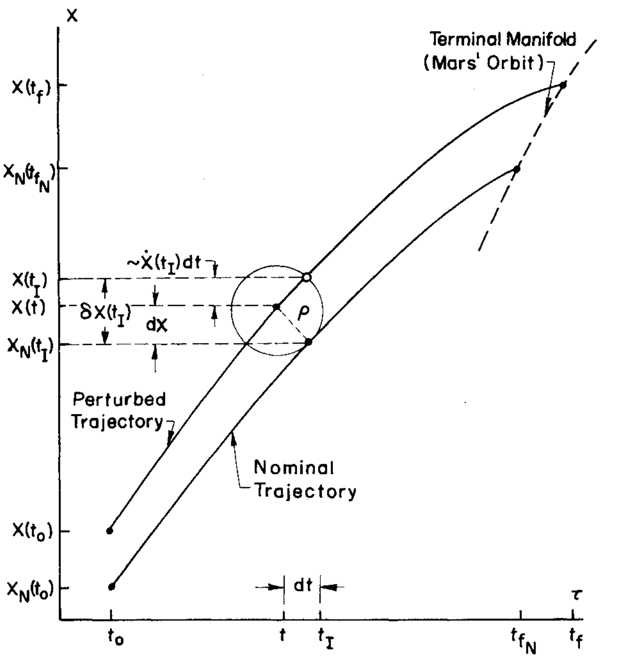

Fig. 2 One state component representation of minimum distance indexing.

If one defines

$$dx = x(t) - x_N(t_I) \tag{46}$$

$$dt = t - t_I \tag{47}$$

and makes the first-order approximation

$$dx \cong \delta x(t) + \dot{x}_N(t_I)\,dt \tag{48}$$

then substitution into Eq. (45) with additional first-order approximations yields[3,5]

$$d(t_f - t) \cong -m_*^T(t_I)\,dx - [1 - m_*^T(t_I) f_N(t_I)]\,dt \tag{49}$$

The index time t_I is chosen so as to make $d(t_f - t)$ in Eq. (49) equal to zero.[3,5] Note that, for a minimum time problem, $m_*(t)$ is identically equal to $-\lambda_N(t)$.[5]

If one defines

$$du = u(t) - u_N(t_I) \tag{50}$$

and makes the first-order approximation

$$du \cong \delta u(t) + \dot{u}_N(t_I)\,dt \tag{51}$$

then combining Eq.. (28), (48), and (51) yields the first-order approximation

$$du \cong -K_*(t_I)\,dx + [\dot{u}_N(t_I) + K_*(t_I) f_N(t_I)]\,dt \tag{52}$$

where the feedback gain matrix is given by

$$K_* = H_{uu}^{-1}(H_{ux} + f_u^T S_*) \tag{53}$$

Thus, the implementation of the time-to-go guidance algorithm at any particular time t may be summarized as follows[3,5]:

1) Guess t_I.

2) Evaluate dx, dt, and $d(t_f - t)$ by means of Eq. (46), (47), and (49), using stored values of $x_N(t)$, $m_*(t)$, and $f_N(t)$, in conjunction with an interpolation scheme.

3) If $d(t_f - t) > (<) 0$, decrease (increase) t_I. Return to step 2. Continue this procedure until $d(t_f - t)$ has been made satisfactorily small or until either end of the nominal trajectory has been reached.

4) Using stored information, in conjunction with an interpolation scheme, evaluate $x_N(t_I)$, $u_N(t_I)$, $K_*(t_I)$, and $\dot{u}_N(t_I) + K_*(t_I) f_N(t_I)$.

5) Determine the control $u(t)$ from the relation

$$u(t) = u_N(t_I) + du \tag{54}$$

with the use of Eqs. (46), (47), and (52).

Min-Distance Indexing

Min-distance (MD) indexing[6,7] differs from TTG indexing in that the index time is chosen to correspond to that point on the nominal trajectory that is at a "minimum distance" from the current point on the perturbed trajectory. This "distance" is actually a metric of weighted position and velocity components and time given by

$$\rho = \sqrt{\sum_{i=1}^{6} a_i [x_i(t) - x_{N_i}(t_I)]^2 + a_0 (t - t_I)^2} \tag{55}$$

where the a_i are positive weighting coefficients, which may be chosen arbitrarily. The index time t_I is that time that minimizes ρ (see Fig. 2).

Minimizing ρ requires that

$$\frac{d(\rho^2)}{dt_I} = -2\sum_{i=1}^{6} a_i [x_i(t) - x_{N_i}(t_I)]\dot{x}_{N_i}(t_I) - 2a_0(t - t_I) = 0 \tag{56}$$

Implementation of the min-distance indexing approach is basically identical to that described in the previous section for time-to-go indexing, except for the specification of the a_i and the replacement of the condition

$$d(t_f - t) = 0 \tag{57}$$

with Eq. (56).

Numerical and Conceptual Refinements of the Guidance Algorithms

Since previous studies of this orbit transfer problem[8-10] obtained large terminal errors using both TTG and MD guidance algorithms, a great deal of attention was given to the accurate numerical implementation of these algorithms in the present study. The first decision made was to use the high-fidelity integrator, DODE, mentioned previously, not only to solve for the nominal trajectory, but also to integrate the perturbed trajectories used to check the performance of the guidance schemes. The use of the integrator's many features and capabilities (described in detail in Refs. 12 and 14) added complexity, but the resulting accuracy helped to make possible the substantially improved guidance accuracies reported below.

A second numerical refinement was the use of the high-order interpolator within the DODE integrator package to compute quantities associated with the nominal trajectory at each index time. In particular, interpolation was done on the backward sweep matrices $S(t)$, $R(t)$, and $Q(t)$, or their starred counterparts, rather than on the gain matrix $K_*(t)$ directly. The reason for this was that the integrated quantities, the backward sweep matrices, could be interpolated much more accurately, using information produced by DODE, than could the derived quantity $K_*(t)$. This approach increased the storage requirements and computational effort in the guidance scheme implementation, but greatly enhanced the numerical accuracy of the process.

Care was also taken in both TTG and MD guidance approaches to iterate on the solution of Eq. (56) or (57) until an

Table 1 **Nondimensionalized terminal state erors for the current and previous studies with perturbations in x_I**

Study		$\delta x_I(0)$ (dimensionless) −0.23-2[a]	−0.58-3	0.29-3	0.29-2	0.29-1
This study						
REO	$\|\psi_v\|$	0.13-11	0.93-9	0.14-9	0.15-11	0.35-9
	$\|\psi_r\|$	0.15-13	0.91-11	0.90-12	0.15-13	0.23-11
TTG	$\|\psi_v\|$	0.10-6	0.48-7	0.24.7	0.11-5	
	$\|\psi_r\|$	0.42-5	0.27-6	0.68-7	0.74-5	
MD1	$\|\psi_v\|$	0.10-5	0.52-7	0.14-7	0.29-5	
	$\|\psi_r\|$	0.42-5	0.27-6	0.68-7	0.70-5	
MD3	$\|\psi_v\|$	0.99-6	0.57-7	0.14-7	0.29-5	
	$\|\psi_r\|$	0.42-5	0.27-6	0.68-7	0.70-5	
Manual	$\|\psi_v\|$					0.43-2
TTG	$\|\psi_r\|$					0.77-2
Hart[8]						
TTG	$\|\psi_v\|$					0.51-1
	$\|\psi_r\|$					0.52-1
MMD2	$\|\psi_v\|$					0.20-1
	$\|\psi_r\|$					0.40-1
MMD1	$\|\psi_v\|$			0.15-2	0.93-2	0.18-1
	$\|\psi_r\|$			0.15-2	0.13-1	0.39-1
Lattimore[9]						
MD	$\|\psi_v\|$			0.44-2		
	$\|\psi_r\|$			0.76-2		
Stoker[10]						
TTG	$\|\psi_v\|$	0.39-3	0.27-3	0.40-3	0.26-1	
	$\|\psi_r\|$	0.34-3	0.15-3	0.13-3	0.20-1	
MD	$\|\psi_v\|$	0.93-3	0.44-3	0.58-3	0.43-2	
	$\|\psi_r\|$	0.21-3	0.18-3	0.17-3	0.25-2	

[a] β-N denotes $\beta \times 10^{-N}$.

Table 2 **Nondimensionalized terminal state errors for the current and previous studies with perturbations in x_4**

Study		−0.21-2	−0.3-3	$\delta x_4(0)$, a.u. 0.5-5	0.5-4	0.3-3	0.5-3	0.33-2
This study								
REO	$\|\psi_v\|$	0.48-12	0.12-9	0.16-8	0.20-10	0.44-10	0.19-9	0.32-11
	$\|\psi_r\|$	0.35-14	0.84-12	0.74-11	0.13-12	0.25-12	0.15-11	0.21-13
TTG	$\|\psi_v\|$	0.19-5	0.35-7	0.57-10	0.64-9	0.26-7	0.74-7	0.17-5
	$\|\psi_r\|$	0.59-6	0.67-8	0.25-10	0.20-9	0.68-8	0.19-7	0.12-5
MD1	$\|\psi_v\|$	0.34-6	0.85-8	0.38-10	0.11-8	0.36-7	0.11-6	0.53-5
	$\|\psi_r\|$	0.53-6	0.65-8	0.24-10	0.20-9	0.92-8	0.31-7	0.27-5
MD3	$\|\psi_v\|$	0.34-6	0.73-8	0.58-10	0.12-8	0.34-7	0.11-6	0.48-5
	$\|\psi_r\|$	0.54-6	0.65-8	0.21-10	0.20-9	0.95-8	0.25-7	0.26-5
Hart[8]								
TTG	$\|\psi_v\|$						0.76-4	
	$\|\psi_r\|$						0.18-3	
MMD2	$\|\psi_v\|$						0.38-4	
	$\|\psi_r\|$						0.78-4	
MMD1	$\|\psi_v\|$						0.64-5	
	$\|\psi_r\|$						0.65-5	
Lattimore[9]								
MD	$\|\psi_v\|$			0.41-2				
	$\|\psi_r\|$			0.68-2				
Stoker[10]								
TTG	$\|\psi_v\|$	0.24-3	0.35-3			0.42-3		0.76-3
	$\|\psi_r\|$	0.15-3	0.16-3			0.13-3		0.13-3
MD	$\|\psi_v\|$	0.58-3	0.35-3			0.41-3		0.81-3
	$\|\psi_r\|$	0.24-3	0.10-3			0.14-3		0.15-3

accurate solution was obtained. This iterative approach is superior to the linear interpolation technique used in Refs. 4 and 5, for example.

Another very useful enhancement, for trajectory simulation purposes, was the switchover from closed-loop state perturbation feedback control, as described by various equations above, to open-loop control near the end of the flight. Equations (29) and (37) reveal that $S_*(t_{f_N})$ does not exist, due to the singularity of $Q(t_{f_N})$, for this problem. Consequently, the feedback gains given by Eq. (53) are divergent at t_{f_N}, which makes the implementation of Eq. (52) difficult as t_I nears t_{f_N}.

This problem may be circumvented by combining Eqs. (27), (48), and (51) to yield

$$du \cong -K(t_I)dx + [\dot{u}_N(t_I) + K(t_I)f_N(t_I)]dt - H_{uu}^{-1} f_u^T R d\bar{\nu} \tag{58}$$

where

$$K = H_{uu}^{-1}(H_{ux} + f_u^T S) \tag{59}$$

and determining $d\bar{\nu}$ by evaluating Eq. (26) at some fixed time $t_{\bar{\nu}} < t_{f_N}$. This control law is effectively an open-loop control law for $t > t_{\bar{\nu}}$, since $d\bar{\nu}$ is determined in terms of $\delta x(t_{\bar{\nu}})$ instead of $\delta x(t)$.

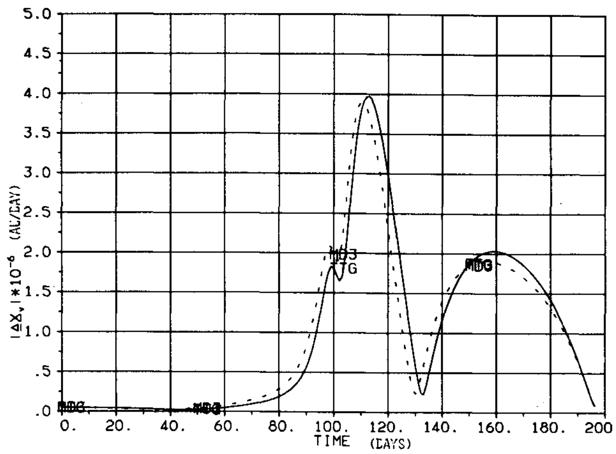

Fig. 3 Velocity deviations, $\delta x_f(0) = -0.23 \times 10^{-2}$ dimensionless velocity units.

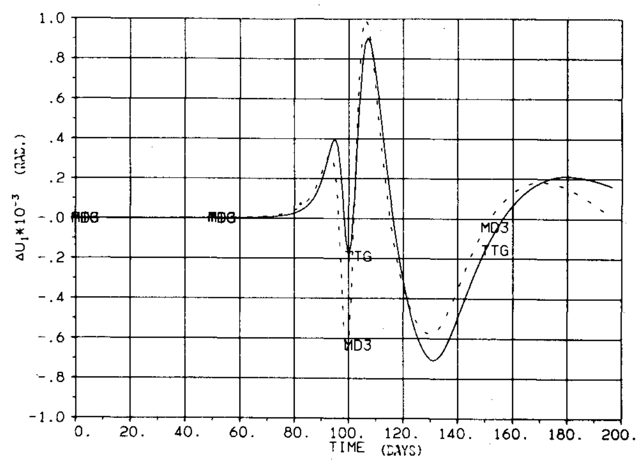

Fig. 5 Out-of-plane control angle deviations, $\delta x_f(0) = -0.23 \times 10^{-2}$ dimensionless velocity units.

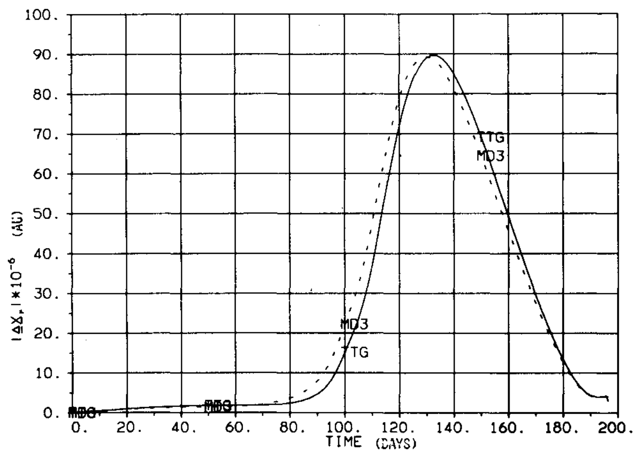

Fig. 4 Position deviations, $\delta x_f(0) = -0.23 \times 10^{-2}$ dimensionless velocity units.

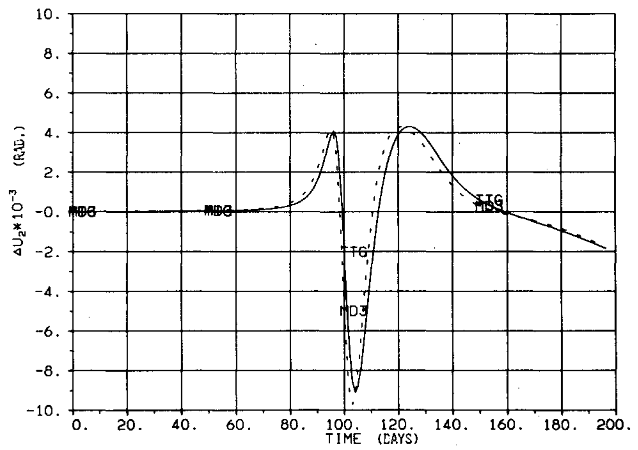

Fig. 6 In-plane control angle deviations, $\delta x_f(0) = -0.23 \times 10^{-2}$ dimensionless velocity units.

The advantage of this alternative expression is that it is well behaved as t nears t_{fN}. The time, $t_{\bar{p}}$, of the switchover from closed- to open-loop guidance is thus an additional input parameter to the guidance scheme.

The appropriate selection of $t_{\bar{p}}$ may be considered from both theoretical and practical viewpoints. For a purely deterministic, perfectly modeled system, as has been mathematically described above, the selection of this time hinges largely on numerical considerations. The opening of the control loop makes it impossible to subsequently compensate for trajectory optimization errors arising from nonlinear terms in the state equations and terminal constraints as well as nonquadratic terms in the performance index, which are neglected in the linearized perturbation theory described above. Moreover, certain numerical roundoff and truncation errors will propagate unchecked after this time. On the other hand, certain roundoff and truncation errors associated with very large (and rapidly varying) gains are avoided after the control loop is opened. Thus, from a theoretical point of view, there is an optimum time to open the control loop (which may vary from one situation to another) that balances these various factors. From a practical point of view, such a procedure might not work well in applications in which the vehicle and, if applicable, the target body motions cannot be predicted precisely. For the purposes of this study, however, the procedure is a reasonable one.

It should be noted that the TTG and MD guidance implementations used by Hart[8] were open-loop subsequent to the initial state perturbation. Lattimore[9] extended Hart's work with MD guidance by periodically closing the guidance loop once per day to once every 30 days. Stoker employed constant feedback gains for the last few days. The TTG implementations of Speyer and Bryson[3] and Wood[4,5] were closed loop until the final time.

A further accuracy enhancement in the present study was the inclusion of certain easily computed second-order terms in Eq. (48). More complicated second-order terms elsewhere were not included. Much more detail about the implementation of the TTG and MD guidance algorithms may be found in Ref. 12.

Numerical Results

The top portions of Tables 1 and 2 present the terminal errors for various magnitudes of initial perturbations studied, using reoptimization (REO), time-to-go guidance (TTG), minimum-distance guidance with weight 1 (MD1), and minimum-distance guidance with weight 3 (MD3). Weight 1 employs weighting coefficients a_i which are inverse squares of Earth's characteristic velocity, characteristic position (1 a.u.), and characteristic time (reciprocal of mean orbital rate). Weight 3, which was used in the previous studies, has only the position components weighted, i.e., $a_{0\text{-}3}=0$, $a_{4\text{-}6}=1$. $|\psi_v|$ and $|\psi_r|$ denote terminal errors in velocity and position in nondimensionalized form. Positions and velocities are made dimensionless by measuring them in units of characteristic position and velocity, as defined above. The tables show that the perturbation guidance schemes exhibit very good behavior as measured by terminal error, especially for the smaller initial perturbations. It is believed that the case $\delta x_I(0)=0.29\times10^{-1}$ dimensionless velocity units lies outside of the range of linearity for perturbation guidance. It was found that a complete trajectory reoptimization produces the lowest final time (it minimizes the performance index the best), although the guidance schemes differ by less than a second (out of 196.75020696 days) for the smaller initial state perturbations, and in no case (except for the large perturbation $\delta x_I(0)=0.29\times10^{-1}$ dimensionless velocity units) is the difference more than 2 min.

Neither of the perturbation guidance schemes exhibits a clear advantage over the other. The two variations in weighting coefficients for minimum distance guidance produce almost identical results. Figures 3-6 are plots of the devaitions (relative to the reoptimized trajectory) in velocity, position, out-of-plane control angle, and in-plane control angle for the TTG and MD3 guidance schemes, with $\delta x_I(0)=-0.23\times10^{-2}$ dimensionless velocity units (-0.4×10^{-4} a.u./day). Notice that the TTG and MD3 schemes are quite similar in behavior. Simulations with other initial conditions also exhibit this similarity. The performances of MD1 and MD3 are even more alike.

Various values of t_p were tried for each case of the current study. The entries in the tables correspond, in each case, to the value of t_p that produced the smallest terminal errors.

Tables 1 and 2 also present the nondimensionalized terminal errors in velocity and position for the previous studies of Hart,[8] Lattimore,[9] and Stoker[10] (MMD1, MMD2, MD, MD1, and MD3 are all forms of min-distance guidance). The performance of the perturbation guidance schemes, as determined in the previous studies, is clearly shown to be inferior to the performance as determined in the current study. Except for the case $\delta x_I(0)=0.29\times10^{-1}$, the improvement in terminal state errors obtained in the present study, relative to those obtained by Hart and Stoker, ranges from about two to five orders of magnitude. The improvement relative to Lattimore's results is even more substantial.

Tables 3 and 4 present the data from Tables 1 and 2 in a more easily interpreted format. Each entry in Tables 3 and 4 is the ratio of the sum of the nondimensionalized terminal velocity and position errors to the nondimensionalized initial perturbation, i.e., $(|\psi_v|+|\psi_r|)/|\delta x(0)|$. The poor showing of the guidance schemes found in the earlier studies can be described in many cases as divergence, i.e., the state perturbations increase rather than decrease with time. This divergence is plainly not evident in the current study.

Also included in Tables 1 and 3 is a result of the present study for the case $\delta x_I(0)=0.29\times10^{-1}$, using a variation of time-to-go indexing in which a shift in final time was calculated off-line and manually input to the guidance algorithm. The reason for the adoption of this procedure was that the standard TTG algorithm did not work for this large an initial perturbation. A limit was placed on the changes in control variables in this case to prevent unrealistically large angular changes (i.e., shifts through several quadrants). The results were not particularly good, but were about an order of magnitude better than those of Hart for the same initial perturbation. In the simpler planar orbit transfer problem of Refs. 4 and 5, it was found that initial state perturbations as large as 0.1 a.u. in position and 0.05 dimensionless velocity units could be handled satisfactorily using time-to-go guidance. The allowable perturbations seem to be substantially smaller in the problem investigated here, perhaps due to the inclusion of three additional terminal constraints (particularly the constraint on final heliocentric angle).

A number of trajectory simulations were also made with current time used as the index time. Whereas previous studies

Table 3 Ratios of terminal errors to initial perturbations in x_I

Study	$\delta x_I(0)$ (dimensionless) -0.23-2	-0.58-3	0.29-3	0.29-2	0.29-1
Hart[8]					
TTG					3.6
MMD2					2.1
MMD1			10.	7.6	2.0
Lattimore[a]			41.		
Stoker[10]					
TTG	0.32	0.72	1.8	16.	
MD	0.50	1.1	2.6	2.3	
This study					
TTG	0.18-2	0.55-3	0.32-3	0.29-2	
MD1	0.23-2	0.56-2	0.28-3	0.34-2	
MD3	0.23-2	0.56-3	0.28-3	0.34-2	
Manual TTG					0.38

Table 4 Ratios of terminal errors to initial perturbations in x_4

Study	δx_4 (0), a.u. -0.21-2	-0.3-3	0.5-5	0.5-4	0.3-3	0.5-3	0.33-2
Hart[8]							
TTG						0.51	
MMD2						0.23	
MMD1						0.26-1	
Lattimore[9]			0.22+4				
Stoker[10]							
TTG	0.19	1.7			1.8		0.27
MD	0.39	1.5			1.8		0.29
This study							
TTG	0.12-2	0.14-3	0.16-4	0.17-4	0.11-3	0.19-3	0.88-3
MD1	0.41-3	0.50-4	0.12-4	0.26-4	0.15-3	0.28-3	0.24-2
MD3	0.42-3	0.46-4	0.16-4	0.28-4	0.15-3	0.27-3	0.22-2

have found this scheme to perform poorly relative to time-to-go or min-distance indexing, the results obtained in the present study were reasonably comparable to those obtained using the other schemes. It should be noted, however, that the current-time indexing scheme was operated open loop near the end of the trajectory (as was done here also for TTG and MD indexing), which was not tried in previous studies. Thus, the potential problem of gain divergence before the end of the flight was avoided. When this scheme was implemented, the index time was never allowed to exceed the nominal final time in those situations in which the actual final time was larger than the nominal final time.

Further information about the trajectory simulations carried out using the various guidance schemes may be found in Ref. 12.

Conclusions

Time-to-go and min-distance indexing schemes for use in neighboring optimum guidance have been re-examined in the context of a low-thrust orbit transfer problem. The two schemes have been shown to work quite well for a range of initial trajectory perturbations and have been found to be generally comparable in performance. These conclusions differ substantially from those of previous studies of the same orbit transfer problem, in which time-to-go indexing was judged to be inferior to min-distance indexing, with nether scheme performing very well. The conclusion that time-to-go indexing works quite well is consistent, however, with certain studies of simpler problems.

A number of numerical refinements were made in the implementation of the guidance schemes, which may have accounted for at least a portion of the substantial improvement in terminal position and velocity errors achieved relative to previously reported results. In addition, the use of a form of open-loop, rather than closed-loop, linearized optimal control logic near the end of the flight appeared to improve the terminal errors. It was also found that current-time gain indexing works reasonably well, at least for the range of perturbations studied, provided that open-loop control logic is used near the end of the flight.

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